UDC 624.071.3: 534.1

THEORY OF SPATIALLY CURVILINEAR ELASTIC BEAMS

PMM Vol. 43, No. 2, 1979, pp. 374-380

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(Received October 13, 1977)

A geometrically nonlinear theory of spatially curved beams is developed. The theory takes into account the rotational inertia, transverse shear deformations, changes in the form and dimensions of the cross sections, and additional loads which arise during the rotation of the cross sections as the beam is deformed. Variants of the hyperbolic equations are obtained and parabolic approximations constructed. The basic relations and equations of motion of the linear theory of curved beams were studied e. g. in [1 — 5]. Improvements in the accuracy of the results of the linear theory were obtained mainly by taking into account the variability of the contour and the warping of the cross sections [3 —5]

1. Formulation of the problem. We consider a naturally twisted beam of variable cross section $F(s)$, made of an elastic isotropic material with constant mechanical characteristics (F is the area of the cross section and s is the arc length of the axial line of the beam). The methods of supporting the end cross sections are assumed known, and the loading conditions given.

Let us identify three points of the beam, P, P_0 and P^*, where P_0 is the projection of P on the axial line of the beam and P^* is the point to which P is translated in the course of deformation. The radius vectors $\mathbf{r}, \mathbf{r}_0$ and $\mathbf{r}^*$ of the points P, P_0 and P^* emerging from the stationary origin satisfy the relations

$$\mathbf{r}^* = \mathbf{r} + \mathbf{u}, \ \mathbf{r} = \mathbf{r}_0 + \eta\mathbf{n} + \zeta\mathbf{b}, \ \mathbf{u} = u_j\mathbf{e}_j \tag{1.1}$$
$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \equiv \mathbf{t}, \mathbf{n}, \mathbf{b}; \ u_1, u_2, u_3 \equiv u, v, w)$$

Here $\mathbf{u}$ and u_j denote the displacement vector of the point P and its components; $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ are the unit vectors of the tangent, normal and binormal to the axial line of the beam ($\mathbf{t} = d\mathbf{r}_0 / ds$); s, η and ζ are the corresponding coordinates of the point P. Repeated indices denote summation from one to three.

Differentiating (1.1) and utilizing the Serret — Frenet formulas [6], we obtain

$$d\mathbf{r}^* = d\mathbf{r} + d\mathbf{u} \tag{1.2}$$
$$d\mathbf{r} = (\partial\mathbf{r} / \partial s)\, ds + (\partial\mathbf{r} / \partial\eta)\, d\eta + (\partial\mathbf{r} / \partial\zeta)\, d\zeta = [(1 - k\eta)\,\mathbf{t} - \varkappa\zeta\mathbf{n} + \varkappa\eta\mathbf{b}]\, ds + \mathbf{n}\, d\eta + \mathbf{b}\, d\zeta$$
$$d\mathbf{u} = (\partial\mathbf{u} / \partial s)\, ds + (\partial\mathbf{u} / \partial\eta)\, d\eta + (\partial\mathbf{u} / \partial\zeta)\, d\zeta = e_{ij}\mathbf{e}_j d\xi_i$$
$$e_{11} = \partial u - kv, \ e_{12} = \partial v + ku - \varkappa w, \ e_{13} = \partial w + \varkappa v \tag{1.3}$$
$$e_{ij} = \partial u_j / \partial\xi_i \ (i = 2, 3; \ j = 1, 2, 3; \ \xi_1, \xi_2, \xi_3 \equiv s, \eta, \zeta; \ \partial \equiv \partial / \partial s)$$
$$(d\mathbf{r})^2 = g_{ij} d\xi_i d\xi_j, \ (d\mathbf{r}^*)^2 = g_{ij}^* d\xi_i d\xi_j$$
$$g_{11} = (1 - k\eta)^2 + \varkappa^2 (\eta^2 + \zeta^2), \ g_{22} = g_{33} = 1$$

$$g_{12} = g_{21} = -\varkappa\zeta,\ g_{13} = g_{31} = \varkappa\eta,\ g_{23} = g_{32} = 0$$
$$g_{ij}\,(\eta = 0,\ \zeta = 0) = \delta_i^{\,j},\quad g_{ij}^* = g_{ij} + 2\varepsilon_{ij}$$
$$2\varepsilon_{ij} = e_{ij} + e_{ji} + e_{is}e_{js} - (1 + \delta_i^{\,j})\, a_{ij}\,(\eta,\ \zeta)$$
$$a_{1j} = (ke_{j1} - \varkappa e_{j3})\,\eta + \varkappa e_{j2}\zeta\ (j = 1,\ 2,\ 3)$$
$$a_{21} = a_{12}, a_{31} = a_{13}, a_{22} = a_{33} = a_{23} = a_{32} = 0$$

Here e_{ij} are the distortion tensor components; g_{ij} and g_{ij}^* are the components of the metric tensor for the initial and deformed state of the beam; $\delta_i^{\,j}$ is the Kronecker delta; ε_{ij} are the strain tensor components; k and $\varkappa$ are the curvature and torsion of the axial line of the beam. From (1.2) it follows that the coordinate system chosen is triorthogonal only for the points lying on the axial line of the undeformed beam.

Expanding the corresponding functions into power series in η and ζ, we obtain

$$(u_j,\ e_{ij},\ \varepsilon_{ij}) = (u_{j,pq},\ e_{ij,pq},\ \varepsilon_{ij,pq})\ \eta^p\zeta^q \tag{1.4}$$

$$e_{11,pq} = \partial u_{,pq} - kv_{,pq}, e_{13,pq} = \partial w_{,pq} + \varkappa v_{,pq} e_{12,pq} = \partial v_{,pq} + ku_{,pq} - \varkappa w_{,pq} \tag{1.5}$$
$$e_{2j,pq} = (p + 1)\, u_{j,p+1\,q},\ e_{3j,pq} = (q + 1)\, u_{j,p\,q+1}\ (j = 1,\ 2,\ 3)$$
$$2\varepsilon_{ij,pq} = e_{ij,pq} + e_{ji,pq} + e_{is,kl}e_{js,p-k\,q-l} - (1 + \delta_i^{\,j})\, a_{ij,pq}$$
$$a_{1j,pq} = (ke_{j1,p-1\,q} - \varkappa e_{j3,p-1\,q}) + \varkappa e_{j2,p\,q-1}$$
$$a_{21,pq} = a_{12,pq},\ a_{31,pq} = a_{13,pq},\ a_{22,pq} = a_{33,pq} = a_{23,pq} = a_{32,pq} = 0$$

Here the indices p, q denote the summation from 0 to ∞, and k, l from 0 to p and q, respectively. The indices preceeding the comma have the same meaning as in (1.2), (1.3).

The Hooke's Law for the triaxial stress-strain state, can be written in the following dimensionless form:

$$\varepsilon_{ij}^\circ = 2\,\nu\,(\varepsilon_{ij} + \delta_i^{\,j}B\theta)$$
$$(\varepsilon_{ij}^\circ = \sigma_{ij}\,/\,E,\ \nu = 1\,/\,[2\,(1 + \mu)],\ B = \mu\,/\,(1 - 2\,\mu),\ \theta = \varepsilon_{ss})$$

Here σ_{ij} and ε_{ij}° are the physical stresses and their dimensionless analogs, E and μ are the Young's modulus and the Poisson's ratio.

Assuming $(\nu, B) = (\nu, B)_{,pq}\eta^p\zeta^q$ and using (1.5), we obtain

$$\varepsilon_{ij}^\circ = \overset{\circ}{\varepsilon}_{ij,pq}\eta^p\zeta^q$$
$$\overset{\circ}{\varepsilon}_{ij,pq} = 2\,\nu_{,kl}\,(\varepsilon_{ij,p-k\,q-l} + \delta_i^{\,j}B_{,r-k\,s-l}\theta_{,p-r\,q-s})$$
$$(k \leqslant r,\ r \leqslant p,\ l \leqslant s,\ s \leqslant q)$$

For the beams with constant mechanical characteristics over the cross sections and along the axial line, we have

$$\overset{\circ}{\varepsilon}_{ij,pq} = 2\,\nu\,(\varepsilon_{ij,pq} + \delta_i^{\,j}B\theta_{,pq}) \tag{1.6}$$

The internal forces and moments can be determined in dimensionless form as follows:

$$\overset{\circ}{N}_{1k} = \overset{\circ}{N}_{k1} = \frac{1}{EF}\iint\limits_{F(s)} \overset{\circ}{\varepsilon}_{1k}\, dF = \overset{\circ}{\varepsilon}_{1k,\,pq}\, i_{pq}^2\ (k = 1,\ 2,\ 3)$$

$$\overset{\circ}{M}_{11} = \overset{\circ}{M}_t = \frac{1}{EF}\iint\limits_{F(s)} (\overset{\circ}{\varepsilon}_{13}\eta - \overset{\circ}{\varepsilon}_{12}\zeta)\, dF = (\overset{\circ}{\varepsilon}_{13,\,p-1\,q} - \overset{\circ}{\varepsilon}_{12,\,p\,q-1})\, i_{pq}^2$$

$$M^{\circ}_{22} = M^{\circ}_{n} = \frac{1}{EF}\iint\limits_{F(s)} \overset{\circ}{\varepsilon}_{11}\zeta dF = \overset{\circ}{\varepsilon}_{11,\, p\, q-1} j^{2}_{pq}$$

$$M^{\circ}_{33} = M^{\circ}_{b} = -\frac{1}{EF}\iint\limits_{F(s)} \overset{\circ}{\varepsilon}_{11}\eta dF = -\overset{\circ}{\varepsilon}_{11,\, p-1\, q}\, j^{2}_{pq}$$

$$j^{2}_{pq} = I_{pq}/(L^{2}F) = 1/\lambda^{2}_{pq}, \quad I_{pq} = \iint\limits_{F(s)} \eta^{p}\zeta^{q} dF$$

$$j_{00}{}^{2} = 1, \; j_{10}{}^{2} = \eta_0, \; j_{01}{}^{2} = \zeta_0 \; (L = 1)$$

Here $N_{11}^{\circ}, N_{12}^{\circ} = N_{21}^{\circ}, N_{13}^{\circ} = N_{31}^{\circ}$ denote the longitudinal and transverse forces; $M_{11}^{\circ}, M_{22}^{\circ}$ and M_{33}° are the torsional and bending moments; F, I_{pq}, η_0 and ζ_0 denote the area, moments of inertia and the coordinates of the center of gravity of the cross section in the system $(\mathbf{t}, \mathbf{n}, \mathbf{b})$; L, λ_{pq} and j_{pq} are the length, flexibility and stability parameters of the beam.

If the line connecting the centers of gravity of the cross sections is used as the axial line, then $\eta_0 = \zeta_0 = 0$. If in addition the principal axes of the cross section coincide with the axes $\mathbf{n}, \mathbf{b}$, then $j_{11} = 0$. In this case $j_{pq} = 0$, provided that at least one of the numbers p, q is odd.

Next we consider the equations of motion of the beam, with help of the Hamilton — Ostrogradskii principle, which can be written in dimensionless form as follows:

$$S = \int\limits_{t_0}^{t_1} \left[\int\limits_{L} (T - U)\, ds \right] dt \tag{1.7}$$

$$T = \iint\limits_{F(s)} \dot{u}_i \dot{u}_i dF = (j^{2}_{pq} \dot{u}_{i,\, kl} \dot{u}_{i,\, p-k\, q-l}) F(s) \qquad \left(u^{\cdot} \equiv \frac{\partial u}{\partial t}\right)$$

$$U = \frac{1}{2\nu} \iint\limits_{F(s)} \overset{\circ}{\varepsilon}_{ij} \varepsilon_{ij} dF = (j^{2}_{pq} \overset{\circ}{\varepsilon}_{ij,\, kl} \varepsilon_{ij,\, p-k\, q-l}) \frac{F(s)}{2\nu}$$

Here T and U are the dimensionless kinetic and potential energy at the cross section s, the linear quantities are related to the beam length L, the velocities to the speed of sound c ($c^2 = 2\nu/\rho$, $t = cT_0/L$; ρ is the material density and T_0 the physical time).

The functions minimizing the functional (1.7) must satisfy the Euler —Ostrogradskii equations, the latter represented in this case by the equations of motion of the beam. The number of equations is equal to the sum of all coefficients of the displacement series. The general form of these equations is as follows:

$$\partial(\partial T/\partial r)/\partial t = \partial(\partial U/\partial p)/\partial s - \partial U/\partial\varphi \tag{1.8}$$

$$r = \partial\varphi/\partial t, \; p = \partial\varphi/\partial s, \; \varphi \equiv u_{i,pq}$$

and they must be supplemented by a specified number of initial and boundary conditions.

Various variants of the theory of beams can be constructed with the help of the finite power series (1.4) only ($p \leqslant p^*$, $q \leqslant q^*$). The magnitude of the resulting errors in the determination of the functions sought obviously diminishes without bounds as $p^*, q^* \to \infty$. The variants of the equations given below are based on the corresponding power expansions in which the only terms retained are those containing η

and ζ in powers not greater than the first. The second and fourth variant adopt, in addition, specified hypotheses concerning the transverse and shear stresses and deformations.

2. Concrete variants of the theory of curved beams. Variant 1. In constructing this variant we assume the stress-strain state of the beam to be triaxial. We retain in the expansions (1.4) and (1.6) only the terms satisfying the condition $p+q\leqslant 1$. The components of the distortion and deformation tensors and the expressions for the kinetic and potential energy assume the form

$$(u_i, e_{ij}, \varepsilon_{ij}, \varepsilon_{ij}^\circ) = (\ldots)_{,0} + (\ldots)_{,1}\eta + (\ldots)_{,2}\zeta \tag{2.1}$$

$$\begin{aligned}
&e_{11,k} = \partial u_k - kv_k,\ e_{12,k} = \partial v_k + ku_k - \varkappa w_k \qquad (2.2)\\
&e_{13,k} = \partial w_k + \varkappa v_k\ (k = 0, 1, 2)\\
&e_{i1,0} = u_{i-1},\ e_{i2,0} = v_{i-1},\ e_{i3,0} = w_{i-1}\ (i = 2, 3)\\
&e_{ij,k} = 0\ (i = 2, 3;\ j = 1, 2, 3;\ k = 1, 2)\\
&2\,\varepsilon_{ij,0} = e_{ij,0} + e_{ij,0} + e_{is,0}e_{js,0}\ (i, j = 1, 2, 3)\\
&2\,\varepsilon_{1j,k} = 2\,\varepsilon_{j1,k} = e_{1j,k} + e_{j1,k} - (1+\delta_i^j)\,a_{1j,k} + e_{1s,0}e_{js,k} +\\
&\quad e_{1s,k}e_{js,0}\ (j = 1, 2, 3;\ k = 1, 2)\\
&2\,\varepsilon_{ij,k} = 0\ \ (i, j = 2, 3;\ k = 1, 2)\\
&a_{1j,1} = ke_{j1,0} - \varkappa e_{j3,0},\ a_{1j,2} = \varkappa e_{j2,0}\ (j = 1, 2, 3)\\
&T = F(s)\,(V_0^2 + 2\,\eta_0V_0V_1 + 2\,\zeta_0V_0V_2 + j_{20}{}^2V_1^2 + 2\,j_{11}{}^2V_1V_2 +\\
&\quad j_{02}{}^2\,V_2^2)\\
&U = F(s)\,(E_0^2 + 2\,\eta_0E_0E_1 + 2\,\zeta_0E_0E_2 + j_{20}{}^2E_1^2 + 2\,j_{11}{}^2E_1E_2 + j_{02}{}^2E_2^2)\\
&V_pV_q = u_{i,p}{}^{\cdot}u_{i,q}{}^{\cdot},\ E_pE_q = \varepsilon_{ij,p}\varepsilon_{ij,q} + \delta_i^jB\theta_{,p}\theta_{,q}\ (p, q = 0, 1, 2)
\end{aligned}$$

where the repeated dots in (2.1) denote the corresponding components from the brackets in the left-hand side.

Calculating the derivatives of T and U with respect to the corresponding variables we arrive, in accordance with (1.8), at the following system of equations:

$$\begin{aligned}
&f(u) = F_{11} - kf(\varepsilon_{12}) + b_{11} \qquad (2.3)\\
&f(v) = F_{12} + kf(\varepsilon_{11}) - \varkappa f(\varepsilon_{13}) + b_{12}\\
&f(w) = F_{13} + \varkappa f(\varepsilon_{12}) + b_{13}\\
&\varphi(u) = \Phi_{11} - k\varphi(\varepsilon_{12}) - \varepsilon_{21,0}^* - b_{21}\\
&\varphi(v) = \Phi_{12} + k\varphi(\varepsilon_{11}) - \varkappa\varphi(\varepsilon_{13}) - \varepsilon_{22,0}^* - b_{22}\\
&\varphi(w) = \Phi_{13} + \varkappa\varphi(\varepsilon_{12}) - \varepsilon_{23,0}^* - b_{23}\\
&\psi(u) - \Psi_{11} - k\psi(\varepsilon_{12}) - \varepsilon_{31,0}^* - b_{31}\\
&\psi(v) = \Psi_{12} + k\psi(\varepsilon_{11}) - \varkappa\psi(\varepsilon_{13}) - \varepsilon_{32,0}^* - b_{32}\\
&\psi(w) = \Psi_{13} + \varkappa\psi(\varepsilon_{12}) - \varepsilon_{33,0}^* - b_{33}\\
&A_{ij} = F^{-1}\partial\,[Fa(\varepsilon_{ij})];\ A_{ij} \equiv F_{ij},\ \Phi_{ij},\ \Psi_{ij};\ a \equiv f, \varphi, \psi\\
&a(u_j) \equiv a(u_j{}^{\cdot\cdot}),\ a(\varepsilon_{ij}) \equiv a(\varepsilon_{ij}{}^*)\\
&f(x) = x_0 + \eta_0x_1 + \zeta_0x_2,\ \varphi(x) = \eta_0x_0 + j_{20}{}^2x_1 + j_{11}{}^2x_2\\
&\psi(x) = \zeta_0x_0 + j_{11}{}^2x_1 + j_{02}{}^2x_2,\ x_k \equiv (u^{\cdot\cdot}, v^{\cdot\cdot}, w^{\cdot\cdot}, \varepsilon_{ij}{}^*)_{,k}\\
&u_j{}^{\cdot\cdot} \equiv \partial^2u_j/\partial t^2\ \ (i = 1;\ j = 1, 2, 3;\ k = 0, 1, 2)\\
&b_{11} = B_1 - kA_2(\varepsilon_{11}),\ b_{12} = B_2 + kA_1(\varepsilon_{11}) - \varkappa A_3(\varepsilon_{11})
\end{aligned}$$

$$
\begin{aligned}
& b_{13} = B_3 + \varkappa A_2(\varepsilon_{11}) \quad (B_i = F^{-1}\partial\,[FA_i(\varepsilon_{11})]) \\
& b_{jk} = A_k(\varepsilon_{ij}) + b(\varepsilon_{ij})(\delta_i{}^j + e_{ik,0}) \quad (i = 1;\ j = 2, 3;\ k = 1, 2, 3) \\
& A_1(\varepsilon_{ij}) = \mathring{\varepsilon}_{ij,0}\,[\eta_0(e_{11,1} - k) + \zeta_0 e_{11,2}] + j_{20}{}^2 \mathring{\varepsilon}_{ij,1}(e_{11,1} - k) + \\
& \quad j_{11}{}^2\,[\mathring{\varepsilon}_{ij,1} e_{11,2} + \mathring{\varepsilon}_{ij,2}(e_{11,1} - k)] + j_{02}{}^2 \mathring{\varepsilon}_{ij,2} e_{11,2} \\
& A_2(\varepsilon_{ij}) = \mathring{\varepsilon}_{ij,0}\,[\eta_0 e_{12,1} + \zeta_0(e_{12,2} - \varkappa)] + j_{20}{}^2 \mathring{\varepsilon}_{ij,1} e_{12,1} + \\
& \quad j_{11}{}^2\,[\mathring{\varepsilon}_{ij,1}(e_{12,2} - \varkappa) + \mathring{\varepsilon}_{ij,2} e_{12,1}] + j_{02}{}^2 \mathring{\varepsilon}_{ij,2}(e_{12,2} - \varkappa) \\
& A_3(\varepsilon_{ij}) = \mathring{\varepsilon}_{ij,0}\,[\eta_0(e_{13,1} + \varkappa) + \zeta_0 e_{13,2}] + j_{20}{}^2 \mathring{\varepsilon}_{ij,1}(e_{13,1} + \varkappa) + \\
& \quad j_{11}{}^2\,[\mathring{\varepsilon}_{ij,1} e_{13,2} + \mathring{\varepsilon}_{ij,2}(e_{13,1} + \varkappa)] + j_{02}{}^2 \mathring{\varepsilon}_{ij,2} e_{13,2} \\
& b(\varepsilon_{ij}) = \eta_0 \mathring{\varepsilon}_{ij,1} + \zeta_0 \mathring{\varepsilon}_{ij,2}, \quad \varepsilon^*_{ij,k} = \mathring{\varepsilon}_{is,k}(\delta_j{}^s + \mathring{e}_{sj,0}) + \\
& \quad (1 - \delta_k{}^\circ)\, \mathring{\varepsilon}_{is,0}\, e_{sj,k} \\
& N_{1k}{}^\circ = N_{k1}{}^\circ = \mathring{\varepsilon}_{1k,0} + \mathring{\varepsilon}_{1k,1}\eta_0 + \mathring{\varepsilon}_{1k,2}\,\zeta_0 \\
& M_{11}{}^\circ = (\mathring{\varepsilon}_{13,0}\eta_0 - \mathring{\varepsilon}_{12,0}\zeta_0) + (\mathring{\varepsilon}_{13,1} j_{20}{}^2 - \mathring{\varepsilon}_{12,1} j_{11}{}^2) + (\mathring{\varepsilon}_{13,2} j_{11}{}^2 - \\
& \quad \mathring{\varepsilon}_{12,2} j_{02}{}^2) \\
& M_{22}{}^\circ = \mathring{\varepsilon}_{11,0}\zeta_0 + \mathring{\varepsilon}_{11,1} j_{11}{}^2 + \mathring{\varepsilon}_{11,2} j_{02}{}^2, \quad M_{33}{}^\circ = -\mathring{\varepsilon}_{11,0}\eta_0 - \mathring{\varepsilon}_{11,1} j_{20}{}^2 - \\
& \quad \mathring{\varepsilon}_{11,2} j_{11}{}^2
\end{aligned}
$$

The above relations and equations of motion simplify considerably when the principal axes of the cross sections are aligned with **n** and **b** ($j_{11} = 0$) and the line connecting the centers of gravity of the cross sections ($\eta_0 \equiv \zeta_0 \equiv 0$) is taken as the axial line of the beam. When $k \equiv \varkappa \equiv 0$, the corresponding linearized system of equations is identical, to within the notation used, to the equations of [7]. In the present variant the components of the displacement vector have a fully defined physical meaning: u_0, v_0 and w_0 are the linear displacements in the **t**, **n** and **b** directions; u_1 and u_2 denote the angular displacements about the **d** and **n** axes, respectively; $\varphi = (w_1 - v_2)/2$ is the angle of rotation of the cross section about the **t**-axis; the parameter $\xi = (w_2 + v_1)/2$ characterizes the change in the area of the transverse cross section, and the coefficients $\eta_1 = (w_2 - v_1)/2$ and $\eta_2 = (w_1 + v_2)/2$ describe the change in the configuration of the cross section. When further terms are retained in the expansions (2.1), then a variant of the theory of beams taking into account the warping of the cross sections can be constructed.

V a r i a n t 2. Here we consider two versions of the theory of beams. The first version assumes that the stresses $\varepsilon_{22}{}^\circ$, $\varepsilon_{33}{}^\circ$ and $\varepsilon_{23}{}^\circ = \varepsilon_{32}{}^\circ$ are absent. The second version assumes that the deformations ε_{22}, ε_{33} and $\varepsilon_{23} = \varepsilon_{32}$ are absent.

From $\varepsilon_{23}{}^\circ = \varepsilon_{32}{}^\circ = 0$ follows $w_1 = -v_2$. Putting $\varepsilon_{22}{}^\circ = \varepsilon_{33}{}^\circ = 0$ we obtain $\varepsilon_{11}{}^\circ = \varepsilon_{11}$, $\varepsilon_{22} = \varepsilon_{33} = -\mu\varepsilon_{11}$, and from this we have $w_2 = v_1 = -\mu(\partial u_0 - kv_0)$. The beams in question have small transverse dimensions, therefore the expressions for $\varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23} = \varepsilon_{32}$ given here and below retain only the terms linear in e_{ij}.

For the components of the displacement vector we have

$$
\begin{aligned}
& u = u_0 + u_1\eta + u_2\zeta, \quad v = v_0 - \psi\eta - \varphi\zeta, \quad w = w_0 + \varphi\eta - \psi\zeta \\
& \varphi = (w_1 - v_2)/2, \quad \psi = \mu(\partial u_0 - kv_0)
\end{aligned}
\tag{2.4}
$$

and here the change in the form of the cross section is disregarded ($\eta_1 = \eta_2 = 0$). Putting $\varepsilon_{22} = 0$, $\varepsilon_{33} = 0$, $\varepsilon_{23} = \varepsilon_{32} = 0$ we obtain, respectively, $v_1 = 0$, $w_2 = 0$ and $w_1 = -v_2$. Then

$$u = u_0 + u_1\eta + u_2\zeta,\ v = v_0 - \varphi\zeta,\ w = w_0 + \varphi\eta \qquad (2.5)$$

Here neither the changes in the form, nor in the area of the transverse cross section are taken into account $(\eta_1 = \eta_2 = 0,\ \xi = 0)$. The basic elasticity relationships and equations of motion are obtained from (2.4) or (2.5) as before.

V a r i a n t 3. This variant of the theory is based on the following expansions:

$$u = u_0 + u_1\eta + u_2\zeta,\quad v = v_0,\quad w = w_0 \qquad (2.6)$$

The equations of motion and the elasticity relationships can be obtained here either from the relations derived above (putting $v_1 = v_2 = 0,\ w_1 = w_2 = 0$), or directly as in the variants 1 and 2. In particular, for the case $\eta_0 = \zeta_0 = 0,\ j_{11} = 0,\ F = \text{const}$ we have

$$u_0^{\cdot\cdot} = \partial\varepsilon_{11,0}{}^* - k\varepsilon_{12,0}{}^*,\quad v_0^{\cdot\cdot} = \partial\varepsilon_{12,0}{}^* + k\varepsilon_{11,0}{}^* - \varkappa\varepsilon_{13,0}{}^* \qquad (2.7)$$
$$w_0^{\cdot\cdot} = \partial\varepsilon_{13,0}{}^* + \varkappa\varepsilon_{12,0}{}^*,\quad u_1^{\cdot\cdot} = \partial\varepsilon_{11,1}{}^* - k\varepsilon_{12,1}{}^* - \lambda_{20}{}^2\varepsilon_{21,0}{}^*$$
$$u_2^{\cdot\cdot} = \partial\varepsilon_{11,2}{}^* - k\varepsilon_{12,2}{}^* - \lambda_{02}{}^2\varepsilon_{31,0}{}^*$$
$$\varepsilon_{11,k}{}^* = \varepsilon_{11,k}{}^\circ(1 + e_{11,0}) + \varepsilon_{12,k}{}^\circ e_{21,0} + \varepsilon_{13,k}{}^\circ e_{31,0} + (1 - \delta_k{}^\circ)\varepsilon_{11,0}{}^\circ e_{11,k}$$
$$(k = 0,\ 1,\ 2)$$
$$\varepsilon_{i1,0}{}^* = \varepsilon_{i1,0}{}^\circ(1 + e_{11,0}),\quad \varepsilon_{i1,0}{}^\circ = \varepsilon_{1i,0}{}^\circ = 2k_{1i}\varepsilon_{i1,0}\quad (i = 2,3)$$
$$\varepsilon_{1j,k}{}^* = \varepsilon_{1j,k}{}^\circ + \varepsilon_{11,k}{}^\circ e_{1j,0} + (1 - \delta_k{}^\circ)\varepsilon_{11,0}{}^\circ e_{1j,k}\quad (j = 2,3;\ k = 0,\ 1,\ 2)$$

Here we have omitted the terms b_{jk} and assumed that $\varepsilon_{22} = \varepsilon_{33} = 0,\ \varepsilon_{23} = \varepsilon_{32} = 0$; k_{1i} are the correction multipliers taking into account the character of the distribution of the tangential stresses $\varepsilon_{1i,0}{}^\circ = \varepsilon_{i1,0}{}^\circ$ over the beam cross section. The rotation of the cross sections about the t -axis and the variation in the form and area of the cross sections are also disregarded $(\varphi = 0,\ \eta_1 = \eta_2 = 0,\ \xi = 0)$.

V a r i a n t 4. Assuming in (2.6) and (2.7) $u_1 \approx -(\partial v_0 + ku_0 - \varkappa w_0),\ u_2 \approx -(\partial w_0 + \varkappa v_0),\ u_1^{\cdot\cdot} \approx 0,\ u_2^{\cdot\cdot} \approx 0$, we arrive at the "classical" variant of the theory of beams. The physical meaning of the relationships given consists of the fact that in the present case we exclude from our considerations the deformations due to the transverse shears and the rotational inertia. The system of differential equations in the present case is a mixed system, unlike the previous hyperbolic equations. In particular, in the linear approximation the equations of motion assume the form

$$u_0^{\cdot\cdot} = \partial e_{11,0} - kj_{20}{}^2\partial(e_{11,1} - ke_{11,0} + \varkappa e_{13,0}) \qquad (2.8)$$
$$v_0^{\cdot\cdot} = j_{20}{}^2\partial^2(e_{11,1} - ke_{11,0} + \varkappa e_{13,0}) + ke_{11,0} - \varkappa j_{02}{}^2\partial(e_{11,2} - \varkappa e_{12,0})$$
$$(\partial^2 \equiv \partial^2 / \partial s^2)$$
$$w_0^{\cdot\cdot} = j_{02}{}^2\partial^2(e_{11,2} - \varkappa e_{12,0}) + \varkappa j_{20}{}^2\partial(e_{11,1} - ke_{11,0} + \varkappa e_{13,0})$$
$$e_{11,0} = \partial u_0 - kv_0,\quad e_{12,0} = \partial v_0 + ku_0 - \varkappa w_0$$
$$e_{13,0} = \partial w_0 + \varkappa v_0,\quad e_{11,1} = -\partial e_{12,0},\quad e_{11,2} = -\partial e_{13,0}$$

where the first equation is hyperbolic while the second and third equations are parabolic.

N o t e. It is possible, while dealing with particular problems, to disregard the longitudinal displacements $(u_0 \equiv 0)$ or to assume that the axial line of the beam is inextensible $(\partial u_0 - kv_0 \equiv 0)$.

3. E x p a n s i o n o f t h e d i s p l a c e m e n t v e c t o r i n t h e

s t a t i o n a r y c o o r d i n a t e s y s t e m. We find, in some problems of dynamics, that it is preferable to consider the motion of the beam not in the components of the system (ξ_i), but in the components of the stationary rectangular coordinate system $(x_i,\ i = 1, 2, 3)$. We denote these components by $v_{j,n}$ $(j = 1, 2, 3;\ n = 0, 1, 2, \ldots)$.

If the axial line of the beam is described by equation $x_i = x_i\ (s)$, then

$$\mathbf{t} = (\partial x_i)\mathbf{e}_i, \quad \mathbf{n} = (\partial^2 x_i)\mathbf{e}_i \ / \ k, \quad k = (\partial^2 x_i \partial^2 x_i)^{1/2}$$

$$\mathbf{b} = \{\partial\ [(\partial^2 x_i)\ /\ k] + k\partial x_i\}\mathbf{e}_i \ / \ \varkappa, \quad \varkappa = \Delta \ / \ k^2$$

Here $\mathbf{e}_i$ denote the unit vectors of the rectangular system, and the columns of the determinant Δ consist of the vectors $\mathrm{col}\ \{\partial x_i,\ \partial^2 x_i,\ \partial^3 x_i\}$.

The components of the displacement vector $v_{j,n}$ can be written in terms of the components $u_{j,n}$ of the form of a matrix product $\mathbf{C} = \mathbf{AB}$. In particular, in case of the variant 1 the matrices have the following structure:

$$\mathbf{A} = \| a_{ij} \|, \quad \mathbf{B} = \| b_{ij} \|, \quad \mathbf{C} = \| c_{ij} \|$$

$$a_{i1} = \partial x_i, \quad a_{i2} = (\partial^2 x_i)\ /\ k, \quad a_{i3} = \{\partial\ [(\partial^2 x_i)\ /\ k] + k\partial x_i\}\ /\ \varkappa$$

$$b_{i1} = u_{i,0}, \quad b_{i2} = u_{j,1} \quad (j = 4 - i,\ i = 1, 2, 3), \quad b_{33} = u_{3,2}$$

$$b_{i3} = u_{k,2} \quad (k = 3 - i,\ i = 1, 2), \quad c_{ij} = v_{j-1,i}$$

In conclusion, we note that the equations obtained include, as particular cases, the equations of dynamics of the plane curvilinear beams $(\varkappa = 0)$ and of the rectilinear twisted beams $(k = 0,\ \varkappa \neq 0)$, as well as those without initial twist $(k = \varkappa = 0;$ see e.g. [8]).

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Translated by L. K.